Stretch Factor in a Planar Poisson-Delaunay Triangulation with a Large Intensity

Nicolas Chenavier, Olivier Devillers

Applied Probability Trust (23 November 2017)

STRETCH FACTOR IN A PLANAR POISSON-DELAUNAY TRIANGULATION WITH A LARGE INTENSITY

NICOLAS CHENAVIER,* *Université du Littoral Côte d'Opale, LMPA Joseph Liouville*

OLIVIER DEVILLERS,** *Inria, Loria, CNRS, Université de Lorraine*

Abstract

Let $X := X_n \cup \{(0,0),(1,0)\}$, where X_n is a planar Poisson point process of intensity n. We provide a first non-trivial lower bound for the distance between the expected length of the shortest path between $(0,0)$ and $(1,0)$ in the Delaunay triangulation associated with X when the intensity of X_n goes to infinity. Simulations indicate that the correct value is about 1.04. We also prove that the expected length of the so-called upper path converges to $\frac{35}{3\pi^2}$, giving an upper bound for the expected length of the smallest path.

Keywords: Delaunay triangulations, Poisson point processes, Stretch factor

2010 Mathematics Subject Classification: Primary 60D05; 05C80

Secondary 84B41

1. Introduction

Let χ be a locally finite subset in $\mathbb{R}^2$, endowed with its Euclidean norm $\|\cdot\|$, such that each subset of size $n < 3$ is affinely independent and no 4 points lie on a circle. The Delaunay triangulation of χ is the unique triangulation with vertices in χ such that the circumdisk of each triangle contains no point of χ in its interior. The set of its edges is denoted by $\mathrm{Del}(\chi)$ and the graph $(\chi, \mathrm{Del}(\chi))$ is the so-called Delaunay graph associated with χ [21, p. 478]. Delaunay triangulations are a very popular structure in computational geometry [1] and are extensively used in many areas such as surface reconstruction [7] or mesh generation [10].

PARTIALLY SUPPORTED BY ANR BLANC PRESAGE (ANR-11-BS02-003).

* Postal address: 50 rue Ferdinand Buisson, BP 699, 62228 Calais Cedex, France

** Postal address: 615 rue du Jardin Botanique, B.P. 101, 54602 Villers-lès-Nancy cedex, France

In this paper, we investigate several paths in $\mathrm{Del}(\chi)$. By a path $P = P(s,t)$ between two points $s, t \in \chi$, we mean a sequence of segments $[Z_0, Z_1], [Z_1, Z_2], \ldots, [Z_{k-1}, Z_k]$, such that $Z_0 = s$, $Z_k = t$. In particular, we say that P is a path in $\mathrm{Del}(\chi)$ if it is a path such that each segment $[Z_i, Z_{i+1}]$ is an edge in $\mathrm{Del}(\chi)$. The investigation of paths is related to walking strategies which are commonly used to find the triangle containing a query point in a planar triangulation [14] or routing in geometric networks [4]. One of the classical works concerns the so-called straight walk which deals with the set of triangles cut by the line segment $[s,t]$. In this context, Devroye, Lemaire and Moreau [15] consider n points evenly distributed in a convex domain and prove that the expected number of Delaunay edges crossed by a fixed line segment is $O\left(\sqrt{n}\right)$. This result is improved by Bose and Devroye [3] who show that, for n points also evenly distributed in a convex domain, the expectation of the maximal number of intersections between a line and edges of the triangulation is $\Theta(\sqrt{n})$.

Another classical problem dealing with paths in triangulations is the investigation of the stretch factor associated with two nodes $s, t \in \chi$ in $\mathrm{Del}(\chi)$. This quantity is defined as $\frac{\ell(S_{s,t}(\chi))}{\|s-t\|}$, where $\ell\left(S_{s,t}(\chi)\right)$ is the length of the shortest path between the source point s and the target point t. Many upper bounds were established for the stretch factor in the context of finite sets χ, e.g. [16, 19]. The best upper bound established until now for deterministic finite sets χ is due to Xia [22] who proves that the stretch factor is lower than 1.998. For the lower bound, Xia and Zhang [23] find a configuration of points χ such that the stretch factor is greater than 1.5932.

In this paper, we focus on a probabilistic version of the problem by taking a slight modification of the underlying point process. More precisely, we consider a homogeneous Poisson point process X_n of intensity n in $\mathbb{R}^2$. For such an infinite point process, studying the maximum of the stretch factor over any points $s, t \in X_n$ has no real sense. Actually, a configuration which is close to the one considered by Xia and Zhang [23], with a stretch factor close to 1.5932, occurs almost surely somewhere in the plane. Thus we take interest to the case where s and t are fixed and added to the Poisson point process.

Another path in the Poisson-Delaunay triangulation, called the Voronoi path, consisting of the set of nuclei of the Voronoi cells that are crossed by a line, was investigated by Baccelli, Tchoumatchenko and Zuyev. In [2], they proved that this Voronoi path has expected length $\frac{4}{\pi} \simeq 1.27$, giving an upper bound for the expected stretch factor. This result has been improved by introducing shortcuts [12] and generalized to higher dimensions [6].

Contributions Let $X := X_n \cup \{s, t\}$, where X_n is a Poisson point process of intensity n,

$$s = (0,0) \quad \text{and} \quad t = (1,0).$$

The main focus of our paper is to provide bounds for the expectation of the stretch factor between s and t in the Delaunay triangulation $\mathrm{Del}(X)$. The difficulty to obtain a lower bound for $\mathbb{E}\left[\ell\left(S_{s,t}(X)\right)\right]$ comes from the fact that we do not know where the shortest path $S_{s,t}(X)$ is. Our first main result deals with the tail of the distribution of $l(S_{s,t}(X))$.

Theorem 1. *There exists $\delta \geq 7 \cdot 10^{-9}$ such that*

$$\mathbb{P}\left[\ell\left(S_{s,t}(X)\right) \leq 1+\delta\right] = O\left(n^{-\frac{1}{2}}\right).$$

As a consequence of Theorem 1 and Lemma 2 (which proves the existence of the limit), we easily deduce the following result

Corollary 1. *There exists $\delta \geq 7 \cdot 10^{-9}$ such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\ell\left(S_{s,t}(X)\right)\right] \geq 1+\delta.$$

We think that our results provide the first non-trivial lower bound (i.e. strictly greater than 1) for the stretch factor when the intensity of the underlying Poisson point process goes to infinity. However, our lower bound is far from optimal since simulations suggest that $\lim_{n\to\infty} \mathbb{E}\left[\ell\left(S_{s,t}(X)\right)\right] \simeq 1.04$. We notice that our result is closely related to a theorem recently proved by Hirsch, Neuhuser, and Schmidt [18, Theorem 26]. Indeed, they show that $\inf_{n\geq 1} \mathbb{E}\left[l(S_{s,t}(X))\right] > 1$. However, their technique cannot provide an explicit lower bound for the stretch factor. In the following proposition, we also give an upper bound.

Theorem 2. *With the above notation, we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\ell\left(S_{s,t}(X)\right)\right] \leq \frac{35}{3\pi^2} \simeq 1.182.$$

The upper bound we considered above is established by bounding the length of a particular path in the Delaunay triangulation. In particular, our theorem provides a more precise upper bound for the expectation of the stretch factor than the one obtained by using the Voronoi path, in which case it was proved that the expected stretch factor is lower than $\frac{4}{\pi} \simeq 1.27$.

2. Preliminaries

Notation Let $s = (0, 0)$ and $t = (k, 0)$ for some integer k. In this section, as in the previous one and the next one, we let $k = 1$. In Section 4, the integer k is arbitrary. Let X_n be a homogenous Poisson point process of intensity n in $\mathbb{R}^2$ and $X = X_n \cup \{s, t\}$. We denote by $\mathrm{Del}(X)$ the Delaunay triangulation associated with X.The following notation will be used throughout the paper.

- For any point $p \in \mathbb{R}^2$, we write $p = (x_p, y_p)$.
- For any segment $e \subset \mathbb{R}^2$, we denote by $h(e) \geq 0$ and $|\widehat{e}| \in [0, \frac{\pi}{2}]$ the length of the horizontal projection of e and the absolute value of the angle with the x-axis respectively.
- For any points $p_1, \ldots, p_k \in \mathbb{R}^2$, we write $p_{1..k} := (p_1, \ldots, p_k)$. When $p_1, \ldots, p_k \in \mathbb{R}^2$ are pairwise distinct, we write the k-tuple of points as $p_{1 \overset{\neq}{.} . k}$. Such a notation will be used in the summation index. Moreover, with a slight abuse of notation, we write $\{p_{1 \overset{\neq}{.} . k}\} := \{p_1, \ldots, p_k\}$.
- For each 3-tuple of points $p_{1 \overset{\neq}{.} . 3} \in (\mathbb{R}^2)^3$ which do not belong to the same line, we denote by $\Delta(p_{1..3})$ and $B(p_{1..3})$ the triangle spanned by $p_{1..3}$ and the (open) circumdisk associated with $p_{1..3}$ respectively. We also denote by $R(p_{1..3})$ the radius of $B(p_{1..3})$.
- For each $z \in \mathbb{R}^2$ and $r \geq 0$, let $B(z, r)$ be the (open) disk centered at z with radius r.
- Let $\mathbb{S} \subset \mathbb{R}^2$ be the unit circle and let σ be the uniform distribution on $\mathbb{S}$ such that $\sigma(\mathbb{S}) = 2\pi$.
- For any Borel subset $B \subset \mathbb{R}^2$, let $\mathcal{A}(B)$ be the area of B. In particular, for each

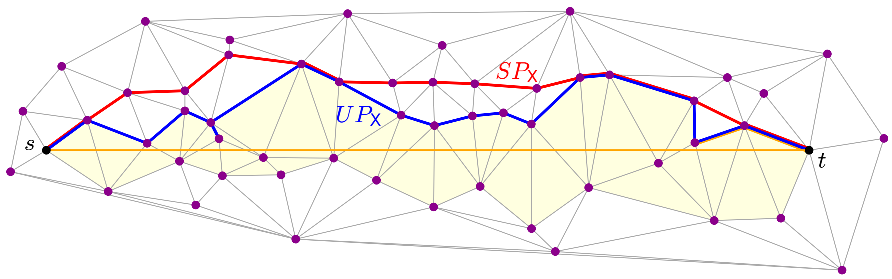

FIGURE 1: The paths $S_{s,t}(X)$ and $U_{s,t}(X)$.

$u_{1..3} \in \mathbb{S}^3$, we have

$$\mathcal{A}\left(\Delta(u_{1..3})\right) = \frac{1}{2}\left|\det\begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix}\right|,$$

where $\beta_i \in [0, 2\pi)$ is the angle between u_i and $(1, 0)$, with $1 \leq i \leq 3$.

Outline In Section 2, we begin with some preliminaries. In Section 3, we provide estimates for the length of a particular path. These estimates directly imply Theorem 2. Section 4 constitutes the main part of our paper and deals with the lower bound for the shortest path. Our main idea is to discretize the plane into pixels and to consider the so-called lattice animals. We derive Theorem 1 by investigating the size of these lattice animals and by adapting tools of percolation theory. A table of integrals which are used throughout the paper is provided in the appendix.

Paths Given $s, t \in \mathbb{R} \times \{0\}$, we denote by $\mathcal{P}_{s,t}(X)$ the family of paths in $\mathrm{Del}(X)$ between the points s and t. For each path $P \in \mathcal{P}_{s,t}(X)$, we denote by $\ell(P)$ the Euclidean length of P and $\mathrm{Card}\,(P)$ its number of edges. We introduce two types of paths in $\mathcal{P}_{s,t}(X)$, namely $S_{s,t}(X)$ and $U_{s,t}(X)$ as follows:

Shortest Path $S_{s,t}(X)$*:* this path minimizes the length between s and t in the Delaunay triangulation $\mathrm{Del}(X)$. Notice that such a path is a.s. unique.

Upper Path $U_{s,t}(X)$*:* this path is defined as the sequence of all edges in $\mathbb{R} \times \mathbb{R}_+$ which belong to Delaunay triangles that intersects $[s, t]$. Some of these edges may be traversed in both ways (e.g. this is the case for one of the edges incident to the fifth vertex of the blue path in Figure 1).

These two paths are depicted in Figure 1.

Sum of the Lengths for a Typical Vertex The following lemma will be used to derive Proposition 1.

Lemma 1. *Let X_n be a Poisson point process of intensity n and let L_s be the sum of the lengths of the edges with vertex $s = (0,0)$ in $\mathrm{Del}(X_n \cup \{s\})$. Then $\mathbb{E}[L_s] = c \cdot n^{-\frac{1}{2}}$ for some constant c.*

In the sense of the Palm theory, L_s is the sum of the lengths of the Delaunay edges starting from a typical vertex.

Proof. Due to the scaling invariance property of a homogeneous Poisson point process, we can assume that $n = 1$. The fact that $\mathbb{E}[L_s]$ is finite is a consequence of the edge-star intensity-relationship derived in [21, Ch10.1].

Notice that the constant c can be made explicit by applying the Slivnyak-Mecke formula (see e.g. [21, Theorem 3.3.5]) and an analogous version of the Blaschke-Petkantschin type change of variables (see e.g. [21, Theorem 7.3.1]) in which one of the vertices is held fixed.

Existence of the limit

Lemma 2. *The expected length of the shortest path $\mathbb{E}[\ell(S_{s,t}(X))]$ between s and t converges as the intensity of X_n goes to infinity.*

Proof. By rescaling, it is enough to prove that for each $n > 0$ the function $k \mapsto \mathbb{E}\left[\ell\left(S_{s,(k,0)}(X)\right)\right]$, with $k \in \mathbb{R}_+$, converges as k goes to infinity. To do it, we introduce a slight modification of $S_{s,(k,0)}(X)$ as follows. Let $\Sigma_{s,(k,0)}(X_n)$ be the shortest path from the nearest neighbor of s in $\mathrm{Del}(X_n)$ to the nearest neighbor of $(k,0)$ in $\mathrm{Del}(X_n)$.

The function $k \mapsto \mathbb{E}\left[\ell\left(\Sigma_{s,(k,0)}(X_n)\right)\right]$ is clearly subadditive. It follows from Fekete's Subadditive Lemma that $\lim_{k\to\infty} k^{-1} \cdot \mathbb{E}\left[\ell\left(\Sigma_{s,(k,0)}(X_n)\right)\right]$ exists. Moreover, according to Lemma 1, we know that $\mathbb{E}\left[\left|\ell\left(S_{s,(k,0)}(X)\right) - \ell\left(\Sigma_{s,(k,0)}(X_n)\right)\right|\right] \leq 2c \cdot n^{-\frac{1}{2}}$. This concludes the proof of Lemma 2.

Remark 1. In the proof of the above result, the main idea was to apply Fekete's Subadditive Lemma. We introduced the auxiliary path $\Sigma_{s,(k,0)}(X_n)$ in such a way that

$k \mapsto \mathbb{E}\left[\ell\left(\Sigma_{s,(k,0)}(X_n)\right)\right]$ is subadditive. This new path was needed because it is not trivial to prove that $k \mapsto \mathbb{E}\left[\ell\left(S_{s,(k,0)}(X_n)\right)\right]$ is also subadditive. Actually, conditioning by the fact that an intermediate point $(x, 0)$, with $0 < x < k$, belongs to the shortest path can increase or decrease the length of the shortest path.

3. Length of the Upper Path

In this section, we estimate the expectation and the variance of the length of the upper path $U_{s,t}(X)$. The following proposition deals with the expectation.

Proposition 1. *Let X_n be a Poisson point process of intensity n. Then*

$$\mathbb{E}\left[\ell\left(U_{s,t}(X)\right)\right] = \frac{35}{3\pi^2} + O\left(n^{-\frac{1}{2}}\right) \simeq 1.182.$$

The above result provides an upper bound for the expectation of the length of the shortest path and implies directly Theorem 2.

Proof. Let

$$L_{X_n} := \sum_{p_{1 \neq 3} \in X_n^3} l_{X_n}(p_{1..3}), \tag{1}$$

where $l_{X_n}(p_{1..3})$ is the half of the length of an edge crossing the line segment $[s, t]$:

$$l_{X_n}(p_{1..3}) = \tfrac{1}{2}\mathbf{1}_{[\Delta(p_{1..3})\in \mathrm{Del}(X_n)]}\mathbf{1}_{[p_{1..3}\in E^+]}\|p_2 - p_1\|$$

and where E^+ is the set of triples of points $p_{1..3}$ such that the triangle spanned by $p_{1..3}$ intersects $[s, t]$ and such that the circumdisk associated with $p_{1..3}$ does neither contain s nor t:

$$E^+ := \{p_{1 \neq 3} \in (\mathbb{R}^2)^3 : \Delta(p_{1..3}) \cap [s, t] \neq \varnothing, B(p_{1..3}) \cap \{s, t\} = \varnothing, y_{p_1}, y_{p_2} > 0, y_{p_3} < 0\}.$$

In the expression of L_{X_n}, we have considered the lengths of the edges in $U_{s,t}(X)$, excepted the ones which contain the points s and t. By Lemma 1, the expected lengths of these two edges is $O\left(n^{-\frac{1}{2}}\right)$. Hence, it is enough to show that $\mathbb{E}\left[L_{X_n}\right] = \frac{35}{3\pi^2} + O\left(n^{-\frac{1}{2}}\right)$. To do it, we apply the Slivnyak-Mecke formula (see e.g. [21, Theorem 3.3.5]). This gives

$$\mathbb{E}\left[L_{X_n}\right] = \tfrac{1}{2}n^3 \int_{\mathbb{R}^6} \mathbb{P}\left[\Delta(p_{1..3}) \cap X_n = \varnothing\right] \mathbf{1}_{[p_{1..3}\in E^+]}\|p_2 - p_1\| \mathrm{d}p_{1..3}.$$

It follows from the Blaschke-Petkantschin type change of variables (see e.g. [21, Theorem 7.3.1]) that

$$\mathbb{E}\left[L_{X_n}\right] = \tfrac{1}{2}n^3\int_{\mathbb{R}_+}\int_{\mathbb{R}^2}\int_{\mathbb{S}^3} e^{-n\pi r^2}\mathbf{1}_{[z+ru_{1..3}\in E^+]}r\|u_2-u_1\| \\ r^3 2\,\mathcal{A}\left(\Delta(u_{1..3})\right)\sigma^3(\mathrm{d}u_{1..3})\mathrm{d}y_z\mathrm{d}x_z\mathrm{d}r, \quad (2)$$

where $z=(x_z,y_z)$. With a slight abuse of notation, we have written $z+ru_{1..3}$ to denote the 3-tuple $(z+ru_1, z+ru_2, z+ru_3)$. Notice that the integrand in the above equation is not 0 only if $z\in[0,1]\times[-r,r]$. To deal with $\mathbf{1}_{[z+ru_{1..3}\in E^+]}$, we introduce an event E'^+ defined as a slight modification of the event E^+. For the event E'^+, we no longer require that the triangle $\Delta(p_{1..3})$ intersects the segment $[s,t]$ but we consider the configuration in which the triangle intersects the supporting line:

$$E'^+ := \{p_{1..3}\in(\mathbb{R}^2)^3 : y_{p_1}, y_{p_2} > 0, y_{p_3} < 0\}.$$

In particular, we have $E^+\subset E'^+$. On the opposite, if $z+ru_{1..3}\in E'^+$ and if $z\in([0,1]\times[-r,r])\setminus B^\cup(s,t,r)$, we have $z+ru_{1..3}\in E^+$, where

$$B^\cup(s,t,r) := \left(B(s,r)\cap\{z : x_z\geq 0\}\right)\cup\left(B(t,r)\cap\{z : x_z\leq 1\}\right).$$

We show below that the right-hand side of (2) is $O(n^{-1/2})$ if the integration over z is restricted to $B^\cup(s,t,r)$. Indeed,

$$\begin{aligned} & n^3\int_0^\infty\int_{B^\cup(s,t,r)}\int_{\mathbb{S}^3} e^{-n\pi r^2}r^4 2\,\mathcal{A}\left(\Delta(u_{1..3})\right)\sigma^3(\mathrm{d}u_{1..3})\mathrm{d}y_z\mathrm{d}x_z\mathrm{d}r \\ & \leq n^3\int_0^\infty \pi r^2\int_{\mathbb{S}^3} e^{-n\pi r^2}r^4 2\,\mathcal{A}\left(\Delta(u_{1..3})\right)\sigma^3(\mathrm{d}u_{1..3})\mathrm{d}y_z\mathrm{d}x_z\mathrm{d}r \\ & \leq \pi n^3\times\frac{15}{16\pi^3 n^{\frac{7}{2}}}\times 24\pi^2 = \frac{45}{2\sqrt{n}} = 22.5\; n^{-\frac{1}{2}}, \end{aligned}$$

where the integrals over r and $u_{1..3}$ are given in Equations (11) and (12) in Appendix A. Replacing E^+ by E'^+ in (2), it follows that:

$$\mathbb{E}\left[L_{X_n}\right] = \tfrac{n^3}{2}\int_0^\infty\int_0^1\int_{-r}^r\int_{\mathbb{S}^3} e^{-n\pi r^2}\mathbf{1}_{[z+ru_{1..3}\in E'^+]}\|u_2-u_1\|r^4 \\ 2\,\mathcal{A}\left(\Delta(u_{1..3})\right)\sigma^3(\mathrm{d}u_{1..3})\mathrm{d}y_z\mathrm{d}x_z\mathrm{d}r + O\left(n^{-\frac{1}{2}}\right).$$

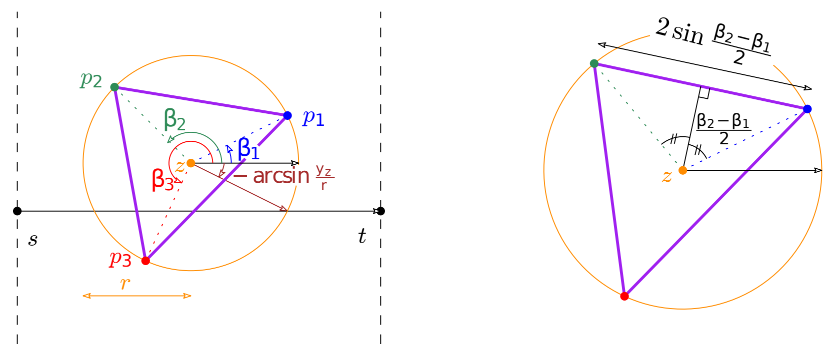

FIGURE 2: Notation for the proof of Proposition 1.

Let $r \geq 0$ be fixed. By taking the change of variables $h = \frac{y_z}{r}$ (see Figure 2), we obtain

$$\begin{aligned}
&\int_{-r}^{r}\int_{\mathbb{S}^3} \mathbf{1}_{[z+ru_{1..3}\in E'^+]}\|u_2-u_1\|\,\mathcal{A}\left(\Delta(u_{1..3})\right)\sigma^3(\mathrm{d}u_{1..3})\mathrm{d}y_z \\
&\quad = r\int_{-1}^{1}\int_{\pi+\arcsin h}^{2\pi-\arcsin h}\int_{-\arcsin h}^{\pi+\arcsin h}\int_{-\arcsin h}^{\beta_2}\det\begin{pmatrix}1 & 1 & 1\\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3\\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3\end{pmatrix} \\
&\qquad\qquad \times 2\sin\frac{\beta_2-\beta_1}{2}\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\beta_3\mathrm{d}h \\
&\quad = \frac{35\pi}{3}r,
\end{aligned}$$

where the last line comes from Equation (13). It follows that

$$\mathbb{E}\left[L_{X_n}\right] = n^3\cdot\frac{35\pi}{3}\int_0^\infty\int_0^1 e^{-n\pi r^2}r^5\mathrm{d}x_z\mathrm{d}r + O\left(n^{-\frac{1}{2}}\right) = \frac{35}{3\pi^2} + O\left(n^{-\frac{1}{2}}\right).$$

The following proposition deals with the variance of the length of the upper path.

Proposition 2. *Let X_n be a Poisson point process of intensity n. Let $\ell\left(U_{s,t}(X)\right)$ be the length of the upper path $U_{s,t}(X)$ in* $\mathrm{Del}(X)$ *from s to t. Then*

$$\mathbb{V}\left[\ell\left(U_{s,t}(X)\right)\right] = O\left(n^{-\frac{1}{2}}\right).$$

Proof. In the same spirit as in the proof of Proposition 1, it is enough to show that $\mathbb{V}\left[L_{X_n}\right] = O\left(n^{-1/2}\right)$. The main idea is to apply a theorem due to Thäle and Yukich on the variance of linear statistics (see Theorem 2.2 in [25]). We first recall their framework, by introducing several definitions and notation.

We say that $A \subset \mathbb{R}^2$ is an admissible set if it is a compact set which is gentle (see p. 2379 in [25] for this few restrictive conditions), regular closed (i.e. such that A equals the closure of its interior) and satisfies $H^2(\partial^2 A) = 0$, where H^2 is the 2-dimensional Hausdorff measure. Let $\xi(p, \chi, \partial A)$ be a translation rotation invariant function defined on triples $(p, \chi, \partial A)$, where $\chi \subset \mathbb{R}^2$ is locally finite, $p \in \chi$ and where $A \subset \mathbb{R}^2$ is an admissible set. The function $\xi(p, \chi, \partial A)$ is referred to as the score function. For shorthand, we write $\xi(p, \chi \cup \{p\}, \partial A)$ as $\xi(p, \chi, \partial A)$ if $p \notin \chi$. Now we state several properties of a score function.

- We say that ξ is *homogeneously stabilizing* if for all $\tau \in (0, \infty)$ and all line L, there is $R := R^\xi(X_\tau, L) \in (0, \infty)$ a.s. (a radius of stabilization), where X_τ is a homogeneous Poisson point process of intensity τ in $\mathbb{R}^2$, such that

$$\xi(o, X_\tau \cap B(o, R), L) = \xi(o, (X_\tau \cap B(o, R)) \cup \mathcal{A}, L) \tag{3}$$

  for all locally finite $\mathcal{A} \subset (B(o, R))^c$.

- We say that ξ is *exponentially stabilizing* with respect to A if for all $p \in \mathbb{R}^2$ there is a radius of stabilization $R := R^\xi(p, P_n) \in (0, \infty)$ a.s. such that

$$\begin{aligned}\xi(n^{1/2}p, &(n^{1/2}P_n) \cap B(p, R), n^{1/2}\partial A)\\ &= \xi\left(n^{1/2}p, \left((n^{1/2}P_n) \cap B(p, R)\right) \cup (n^{1/2}\mathcal{A}), n^{1/2}\partial A\right)\end{aligned}$$

  for all locally finite $\mathcal{A} \subset (B(p, n^{-1/2}R))^c$ and where the tail probability $\tau(t) := \sup_{n>0, p\in\mathbb{R}^2} \mathbb{P}\left[R^\xi(p, P_n) > t\right]$ satisfies $\limsup_{t\to\infty} t^{-1} \log \tau(t) < 0$.

- The score function ξ satisfies the *m-moment condition* with respect to A if there exists $m > 2$ and there is a bounded integrable function $G^{\xi,m} : \mathbb{R} \to \mathbb{R}_+$ with $\int_{\mathbb{R}} r \left(G^{\xi,m}(r)\right)^{1/m} \mathrm{d}r < \infty$ such that for all $r \in \mathbb{R}$

$$\sup_{z\in\mathbb{R}^2\cup\varnothing} \sup_{p\in\mathcal{R}} \sup_{n>0} \mathbb{E}\left[\left|\xi(n^{1/2}p + ru_p, n^{1/2}(P_n \cup \{z\}), n^{1/2}\partial A))\right|^m\right] \leq G^{\xi,m}(|r|),$$

  where $\mathcal{R}$ is the boundary of $[0, 1]^2$ without its extremal points, and where u_p is the unit outward-pointing normal vector to $\partial[0, 1]^2$ at p.

- The score is *well-approximated by P_n input on half-spaces* if for all admissible set A, almost all $x \in \partial A$ and all $w \in \mathbb{R}^2$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\xi(w, n^{1/2}(P_n - y), n^{1/2}(\partial A - y)) - \xi(w, n^{1/2}(P_n - y), L_y)\right|\right],$$

  where L_y is the line tangent to o at $\partial(A - y)$, with $y \in \partial A$.

We rewrite below a weak version of Theorem 2.2 in [25] in our context.

Theorem 3. *(Thäle & Yukich, 2016) Let P_n be a Poisson point process of intensity n in $[-2,2]^2$. Let $\xi(p,\chi,A)$ be a score function which is homogeneously stabilizing, exponentially stabilizing, satisfies the moment condition for some $m>2$ and is well-approximated by P_n on half-spaces. Then there exists a constant $\sigma^2 \in [0,\infty)$ such that*

$$\lim_{n\to\infty} n^{-1/2}\mathbb{V}\left[\sum_{x\in P_n}\xi(n^{1/2}x, n^{1/2}P_n, n^{1/2}\partial[0,1]^2)\right] = \sigma^2.$$

In Theorem 2.2 in [25], Thäle and Yukich consider a Poisson point process $\mathcal{P}_\lambda := \mathcal{P}_{\lambda\kappa}$ of intensity $\lambda\kappa$ in $[0,1]^d$, $d\geq 2$, with κ a bounded density on $[0,1]^d$, a fixed admissible set $A_0\subset[0,1]^d$ and a score function ξ defined on triples (x,χ,A). By scaling and by translation invariance, we consider a homogeneous Poisson point process P_n of intensity n in $[-2,2]^2$, by letting $A_0 := [0,1]\times[0,-1]$.

Now we show how the above result implies that $\mathbb{V}[L_{X_n}] = O\left(n^{-1/2}\right)$. Let L_{P_n} be defined as L_{X_n} by replacing X_n by P_n in (1). It is enough to prove that $\mathbb{V}[L_{P_n}] = O(n^{-1/2})$. To do it, let $\xi(p,\chi,\partial A)$ be the translation rotation invariant score function defined as follows: if $p\in\chi\cap A^c$ then

$$\xi(p,\chi,\partial A) = \tfrac{1}{2}\sum_{\{p_2,p_3\}\subset P_n}\mathbf{1}_{[\Delta(p,p_2,p_3)\in\mathrm{Del}(P_n)]}\mathbf{1}_{[p_2\notin A, p_3\in A]}\|p_2-p\|, \tag{4}$$

where $\chi\subset\mathbb{R}^2$ is a locally finite set and where A is an admissible set. Otherwise, we let $\xi(p,\chi,\partial A)=0$. In view of limits such as (3), we need to define the score on the line $\mathbb{R}\times\{0\}$. We thus put $\xi(p,\chi,\mathbb{R}\times\{0\})$ as in (4) by replacing the admissible set A by $\mathbb{R}\times\mathbb{R}_-$ and its boundary ∂A by $\mathbb{R}\times\{0\}$.

Let $S_n := \sum_{p\in P_n}\xi(p,P_n,\partial A_0)$, where we recall that $A_0=[0,1]\times[-1,0]$. Hence S_n is the length of the (closed) path in $\mathrm{Del}(P_n)\cap([0,1]^2)^c$ associated with the Delaunay triangles intersecting the boundary of A_0. We show below that the score $\chi(p,\chi,\partial A)$ satisfies the four assumptions of Theorem 3. This will be sketched because many similar examples were already derived from Theorem 3, e.g. Theorems 1.1-1.5 in [25] (see also four applications in Section 2 in [24] which are derived from a result which is very similar to Theorem 3).

For the first assumption, let L be a line and recall that X_τ is a homogeneous Poisson point process of intensity τ. Equation (3) is satisfied by letting $R^\xi(X_\tau, L)$

as twice the radius of the smallest disk containing the Voronoi cell with nucleus o in the Voronoi tessellation associated with the point process $X_\tau \cup \{0\}$. Indeed, adding a point outside $B(o, R^\xi(X_\tau, L))$ does not modify the Delaunay edges with vertex o. This proves that ξ is homogeneously stabilizing. In the same spirit, we can show that ξ is exponentially stabilizing since the tail of the distribution of the circumradius of the typical Voronoi cell converges to 0 at a rate at least exponential (see e.g. Equation (6) in [5]). The fact that ξ is well-approximated by P_n input on half-spaces is trivial because $A_0 = [0,1] \times [0,-1]$. It remains to prove that ξ satisfies the m-moment condition. We use similar arguments as the ones appearing in the proof of Theorem 2.1 in [24].

Let $z \in \mathbb{R}^2$ be fixed and let $p \in \mathcal{R}$. We denote by $L_{z,p,n}$ the sum of the lengths of the Delaunay edges with vertex p in $\mathrm{Del}(n^{1/2}(P_n \cup \{z\}))$. Moreover, we also let $C_{n^{1/2}(P_n\cup\{z\})}(n^{1/2}p + ru_p)$ as the Voronoi cell with nucleus $n^{1/2}p + ru_p$ in the Voronoi tesselation associated with the point process $n^{1/2}(P_n \cup \{z\})$. Assume that the score $\xi(n^{1/2}p+ru_p, n^{1/2}(P_n\cup\{z\}), n^{1/2}\partial A_0)$ is non-zero. In this case there exists a Delaunay edge incident to $n^{1/2}p+ru_p \in (n^{1/2}A_0)^c$ (with length at least r) such that the endpoint belongs to $n^{1/2}A_0$. Hence $2 \cdot C_{n^{1/2}(P_n\cup\{z\})}(n^{1/2}p + ru_p)$ intersects $n^{1/2}A_0$. It follows that

$$\xi(n^{1/2}p+ru_p, n^{1/2}(P_n \cup \{z\}), n^{1/2}\partial A_0) \leq L_{z,p,n}\mathbf{1}_{[2\cdot C_{n^{1/2}(P_n\cup\{z\})}(n^{1/2}p+ru_p)\cap n^{1/2}\partial A_0 \neq \varnothing .]}$$

In the same spirit as Lemma 2, we can prove that $L_{z,p,n}$ has finite moments of all orders, uniformly in p and z. Moreover, it may be seen that the probability of the event $\{C_{n^{1/2}(P_n\cup\{z\})}(n^{1/2}p + ru_p) \cap n^{1/2}\partial A_0 \neq \varnothing\}$ decays exponentially fast in r, uniformly in p and z. The CauchySchwarz inequality ensures that the m-moment condition is satisfied.

As a corollary, we obtain an estimate of the tail of the length of the upper path.

Corollary 2. *With the same notation as above, we have*

$$\mathbb{P}\left[\ell\left(U_{s,t}(X)\right) > 1.2\right] \quad = \quad O\left(n^{-\frac{1}{2}}\right).$$

Proof. It follows from Chebyshev's inequality that

$$\mathbb{P}\left[\ell\left(U_{s,t}(X)\right) > 1.2\right] \leq \frac{\mathbb{V}\left[\ell\left(U_{s,t}(X)\right)\right]}{1.2 - \mathbb{E}\left[\ell\left(U_{s,t}(X)\right)\right]}.$$

This concludes the proof according to Proposition 1 and Proposition 2.

4. Lower Bound on Shortest Path

In this section, we prove Theorem 1. By scaling invariance, our problem is of the same type as if the intensity n is constant and $s = (0,0)$, $t = (k,0)$, where $k \in \mathbb{N}^*$ goes to infinity. The main idea is to discretize the plane $\mathbb{R}^2$ into squares (called pixels) and to define a lattice animal. For each pixel, we introduce a horizontality property. This property holds if there exists at least one path, through the pixel, which is almost horizontal (in a sense which will be specified). Then we prove that there exists a non-negligible proportion of pixels intersected by the smallest path for which the horizontality property does not hold. This will provide a lower bound for the length of the smallest path.

We begin our proof by introducing formally the notions of *pixels*, *animal lattices*, and *horizontality property*. Then we establish auxiliary results which will be the key ingredients to derive Theorem 1. These auxiliary results will be proved at the end of this section.

4.1. Preliminaries

4.1.1. *Pixels and Lattice Animals* Recall that $s = (0,0)$ and $t = (k,0)$ for some integer k. We discretize $\mathbb{R}^2$ into pixels as follows. Let $\mathbf{G} = (\mathbb{Z}^2, E)$ be the graph with set of edges satisfying $(v,w) \in E \Leftrightarrow ||v-w|| = 1$, where $v, w \in \mathbb{Z}^2$. In digital geometry, the graph is known as a *4-connected neighborhood* and each vertex of $\mathbf{G}$ is called a *pixel*. Moreover, for each $v \in \mathbb{Z}^2$, we consider different scaled versions of squares:

$$C(v) := v \oplus [-\tfrac{1}{2}, \tfrac{1}{2}]^2, \quad C^\varepsilon(v) := v \oplus [-\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon]^2; \quad C_\lambda(v) := v \oplus [-\tfrac{\lambda}{2}, \tfrac{\lambda}{2}]^2,$$

where $\oplus$ denotes the Minkowski sum, $\lambda \in \mathbb{N}^*$ and $\varepsilon \in \mathbb{R}_+^*$. With a slight abuse of notation, we also say that $C(v)$ is a pixel.

We define scaled and translated versions of the grid $\mathbf{G}$ as follows. For $\lambda \in \mathbb{N}^*$ and $\tau \in \mathbb{Z}^2$ we denote by $\lambda\mathbf{G}$ the grid of points in $(\lambda\mathbb{Z})^2$ with edges of length λ and $\tau + \lambda\mathbf{G}$ its translation by τ. We also split $\mathbf{G}$ in 4 subgrids as follows: each subgrid is referred to as a color $c \in Colors$, where $Colors := \{green, pink, blue, yellow\}$. Each subgrid with color c is denoted:

$$\mathbf{G}_{(c)} = 2\mathbf{G} + O_{(c)},$$

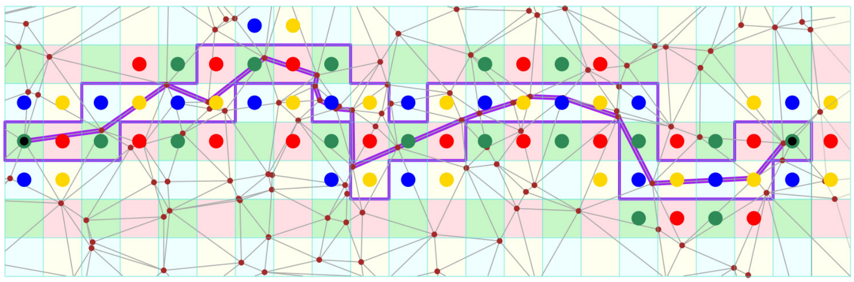

FIGURE 3: Lattice animals. The pixels $C(v)$ for $v \in \mathbf{A}(P)$ are hashed and the path P is purple. The $\mathbf{A}_{(c)}(P)$, for each $c \in Colors$, are referred to as big dots of the relevant color.

where

$$O_{(green)} = (0,0), \quad O_{(pink)} = (1,0), \quad O_{(blue)} = (0,1), \quad \text{and } O_{(yellow)} = (1,1).$$

Notice that when v and w are two pixels with the same color, the interiors of the squares $C_2(v)$ and $C_2(w)$. Such a property will be useful to ensure the independence of a suitable family of random variables.

We introduce below the so-called notion of animal [11]. Given a graph $G' := (V', E')$, a *lattice animal* is a collection of vertices $\mathbf{A} \subset V'$ such that for every pair of distinct vertices $v, w \in \mathbf{A}$ there is a path in G' connecting v, w visiting only vertices in $\mathbf{A}$. With each path P in the Delaunay triangulation, we associate the so-called lattice animal of P in $\mathbf{G}$ (see Figure 3):

$$\mathbf{A}(P) = \{v \text{ vertex of } \mathbf{G} : C(v) \cap P \neq \varnothing\}.$$

In the same spirit as above, for each $c \in Colors$, we let

$$\mathbf{A}_{(c)}(P) = \{v \text{ vertex of } \mathbf{G}_{(c)} : C_2(v) \cap P \neq \varnothing\}$$

We also consider animals at different scales (see Figure 4):

$$\mathbf{A}_{(c),\lambda}(P) = \{v \text{ vertex of } \lambda\mathbf{G} + O_{(c)} : C_\lambda(v) \cap P \neq \varnothing\}.$$

4.1.2. *Properties on Pixels* For any pixel $v \in \mathbb{Z}^2$, we consider three events, namely $\mathcal{I}_\varepsilon(v)$, $\mathcal{H}_\rho(v)$, and $\mathcal{H}'_{\varepsilon,\alpha,\kappa}(v)$. These events depend on four parameters $\varepsilon > 0$, $\rho > 0$, $\alpha \in [0, \frac{\pi}{2}]$, and $\kappa > 1$ and are described below.

- *Independence property* $\mathcal{I}_\varepsilon(v)$: this event holds if, for any Delaunay triangle in $\mathrm{Del}(X)$ intersecting $C^\varepsilon(v)$, the circumdisk of the triangle is included in $C_2(v)$ and $\|s-v\| \geq 2$ and $\|t-v\| \geq 2$ (to avoid pixels that are neighbors of s or t). Notice that this event is $\sigma(X_n \cap C_2(v))$ measurable.
- *Strong horizontality property* $\mathcal{H}_\rho(v)$: this event holds if there exists a path along Delaunay edges in $\mathrm{Del}(X)$, between $x_v - \frac{1}{2}$ and $x_v + \frac{1}{2}$, intersecting $C(v)$ and with length smaller than $1+\rho$. A path satisfying this property is denoted by $\mathcal{PH}_\rho(v)$. The first and the last edges of such a path are clipped by the vertical lines $x = x_v - \frac{1}{2}$ and $x = x_v + \frac{1}{2}$.
- *Weak horizontality property* $\mathcal{H}'_{\varepsilon,\alpha,\kappa}(v)$: this event holds if the total length $L_{\varepsilon,\alpha,\kappa}(v)$ of the horizontal projection of all edges in $\mathrm{Del}(X)$ intersecting $C^\varepsilon(v)$ and having an angle with respect to the x-axis smaller than α, is greater than $\frac{1}{\kappa}$, i.e.
$$L_{\varepsilon,\alpha,\kappa}(v) = \sum_{e, e\cap C^\varepsilon(v)\neq\varnothing, |\widehat{e}|\leq\alpha} h(e) \geq \tfrac{1}{\kappa}.$$

As we will see in Lemma 5, under a suitable choice of $\varepsilon, \alpha, \kappa$, the strong horizontality property implies the weak horizontality property.

4.2. Auxiliary Results

In this section, we establish auxiliary results which will be used to derive Theorem 1. The first one provides a lower bound for the length of any path with respect to the number of pixels with a strong horizontality property.

Proposition 3. *Let $P \in \mathcal{P}_{s,t}(X)$ and $\rho > 0$, $\varepsilon > 0$. Then*
$$\ell(P) \geq k + \rho\left(k - 4 \max_{c\in Colors} \sharp_{\mathcal{H}}(\mathbf{A}_{(c)}(P))\right),$$
where $\sharp_{\mathcal{H}}(\mathbf{A}_{(c)}(P))$ is the number of pixels v in $\mathbf{A}_{(c)}(P)$ such that $\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_\varepsilon(v)$.

As a second auxiliary result, we prove that the number of pixels in $\mathbf{A}(S_{s,t}(X))$ which satisfy a property $\mathcal{Y}$ is not large with high probability. In the sequel, we will use the following notation:
$$\mathbb{Z}^{2,(c)}_{s,t} := \mathbb{Z}^2_{s,t} \cap \{v \in \mathbb{Z}^2 \text{ with color c}\},$$
where
$$\mathbb{Z}^2_{s,t} := \mathbb{Z}^2 \setminus \{v \in \mathbb{Z}^2 : \|v-s\| \leq 2 \text{ or } \|v-t\| \leq 2\}.$$

Proposition 4. *Let $p \in (0, 0.01]$ and let $\mathcal{Y} := (Y_v)_{v\in\mathbb{Z}^2}$ be a family of events such that, for any color c, the events $(Y_v)_{v\in\mathbb{Z}^{2,(c)}_{s,t}}$ are independent and $p = \mathbb{P}[Y_v]$ for each $v \in \mathbb{Z}^{2,(c)}_{s,t}$. For any $\mathbf{A} \subset \mathbb{Z}^2$, we denote by $\sharp_{\mathcal{Y}}(\mathbf{A}) = \sum_{v\in\mathbf{A}} \mathbf{1}_{[Y_v]}$ the number of pixels v in $\mathbf{A}$ such that the event Y_v holds. Then for any color c we have*

$$\mathbb{P}\left[\sharp_{\mathcal{Y}}(\mathbf{A}_{(c)}(S_{s,t}(X))) \geq k\sqrt{p}\right] = O\left(k^{-\frac{1}{2}}\right).$$

This result will be applied to the case where $\mathcal{Y}$ is the strong horizontality property. To apply such a result, we have to estimate the probability of the event $\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)$ for a suitable choice of ε_ρ.The following proposition shows that the probability that a pixel has a (strong) horizontality property is small.

Proposition 5. *Let $v \in \mathbb{Z}^2_{s,t}$ and $0 < \rho < 4 \cdot 10^{-6}$ and let $\varepsilon_\rho := \sqrt{\rho}\sqrt{2+\rho}$. Then*

$$\mathbb{P}\left[\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)\right] \leq P(\rho, n),$$

where

$$P(\rho, n) := 95n^3 e^{-0.194n} + \left(19n^2 + 13n + 4\right)e^{-n\pi} + 31.76\left(\frac{3}{4} + \frac{\sqrt{\rho}\sqrt{\rho+2}}{2}\right)^2 \sqrt{\rho n}.$$

The above result constitutes one of the main difficulties to prove Theorem 1.

4.3. Proof of Theorem 1

Let ρ and n be such that $p \leq 0.01$. According to Proposition 3, we have

$$\begin{aligned}\mathbb{P}\left[\ell\left(S_{s,t}(X)\right) \leq k + \rho\left(k - 4k\sqrt{p}\right)\right] &\leq \mathbb{P}\left[\max_{c\in Colors} \sharp_{\mathcal{H}}(\mathbf{A}_{(c)}(S_{s,t}(X))) \geq k\sqrt{p}\right] \\ &\leq \sum_{c\in Colors} \mathbb{P}\left[\sharp_{\mathcal{H}}(\mathbf{A}_{(c)}(S_{s,t}(X))) \geq k\sqrt{p}\right].\end{aligned}$$

To bound the right-hand side, we apply Proposition 4 to the family of events $\mathcal{Y} := (Y_v)_{v\in\mathbb{Z}^2}$, where $Y_v := \mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)$. Notice that for any color c, the events $(Y_v)_{v\in\mathbb{Z}^{2,(c)}_{s,t}}$ are independent: this comes from the fact that the event $\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)$ is $\sigma(X_n \cap C_2(v))$ measurable and the fact that the interiors of $C_2(v)$ and $C_2(w)$ are disjoint when $v \neq w \in \mathbb{Z}^{2,(c)}_{s,t}$. Moreover, for any $v \in \mathbb{Z}^{2,(c)}_{s,t}$, the probability $p := \mathbb{P}\left[\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)\right]$ does not depend on v. Taking $\rho \leq 4 \cdot 10^{-6}$, we deduce from Proposition 4 and Proposition 5 that

$$\mathbb{P}\left[\ell\left(S_{s,t}(X)\right) \leq k + \rho\left(1 - 4\sqrt{P(\rho, n)}\right)k\right] = O(k^{-1/2}).$$

To get the best lower bound in Theorem 1 we have to maximize $\rho\left(1-4\sqrt{P(\rho,n)}\right)$. The values $\rho=3.4\cdot 10^{-8}$ and $n=136$ yields a good estimate for this bound. Then we have $\rho<4\cdot 10^{-6}$ and $p\leq P(\rho,n)\simeq 0.04<0.01$, so that the assumptions of Propositions 4 and 5 are satisfied. This concludes the proof of Theorem 1 since

$$\left(1-4\sqrt{P(\rho,n)}\right)\geq 7\cdot 10^{-9}.$$

4.4. Proofs of the Auxiliary Results

4.4.1. *Proof of Proposition 3* Splitting the path P into vertical columns, we have $\ell\left(P\right)=\sum_{i\in\mathbb{Z}}\ell\left(P\cap Col[i]\right)$, where, $Col[i]=[i-\frac{1}{2},i+\frac{1}{2}]\times\mathbb{R}$ is the ith column for each $i\in\mathbb{Z}$. For $i\in\{1,\dots,k-1\}$, let $v[i]$ be the lowest pixel of $\mathbf{A}(P)\cap Col[i]$ such that there is a connected component of $P\cap Col[i]$ intersecting $C(v[i])$ and the left and right side of $Col[i]$. Notice that such a pixel exists for each column $1\leq i\leq k-1$. On the event $\neg\mathcal{H}_\rho(v[i])\wedge\mathcal{I}_\varepsilon(v[i])$, we use the fact that $\ell\left(P\cap Col[i]\right)\geq 1+\rho$. On the complement of this event, we use the trivial inequality $\ell\left(P\cap Col[i]\right)\geq 1$. Denoting by $N:=\sum_{i=1}^{k-1}\mathbf{1}_{[\mathcal{H}_\rho(v[i])\vee\neg\mathcal{I}_\varepsilon(v[i])]}$ the number of horizontal pixels on the path, it follows that

$$\begin{aligned}\ell\left(P\right)&\geq\sum_{i=1}^{k-1}\mathbf{1}_{[\mathcal{H}_\rho(v[i])\vee\neg\mathcal{I}_\varepsilon(v[i])]}+\sum_{i=1}^{k-1}(1+\rho)\mathbf{1}_{[\neg\mathcal{H}_\rho(v[i])\wedge\mathcal{I}_\varepsilon(v[i])]}\\&\geq N+(1+\rho)(k-N)=k+\rho(k-N).\end{aligned}$$

Then we conclude the proof by observing that

$$N\leq\sum_{v\in\mathbf{A}(P)}\mathbf{1}_{[\mathcal{H}_\rho(v)\vee\neg\mathcal{I}_\varepsilon(v)]}=\sum_{c\in Colors}\sharp_{\mathcal{H}}(\mathbf{A}_{(c)}(P))\leq 4\max_{c\in Colors}\sharp_{\mathcal{H}}(\mathbf{A}_{(c)}(P)).$$

4.4.2. *Proof of Proposition 4* First, we establish a result which provides an upper bound for the size of the lattice animal of $S_{s,t}(X)$ with high probability.

Lemma 3. *Let $S_{s,t}(X)\in\mathcal{P}_{s,t}(X)$ with $\|s-t\|=k$. For any $\lambda\geq 2$, $c\in Colors$, and $k>0$, let $\mathcal{E}(c,k,\lambda)$ be the event:*

$$\mathcal{E}(c,k,\lambda):=\left\{\mathrm{Card}\left(\mathbf{A}_{(c),\lambda}(S_{s,t}(X))\right)<\tfrac{2.55\,k}{\lambda}+1\right\}. \tag{5}$$

Then $\mathbb{P}\left[\mathcal{E}(c,k,\lambda)\right]=1-O\left(k^{-\frac{1}{2}}\right)$.

Proof. According to a result due to Gerard, Favreau and Vacavant [17], we know that

$$\mathrm{Card}\,(\mathbf{A}(P)) \leq \tfrac{3\sqrt{2}}{2}\ell\,(P) + 1$$

for any $s, t \in \mathbb{R}^2$ and for any path P between s and t, not necessarily in $\mathrm{Del}(X)$ (a more concise proof of the above inequality can be found in [9]). As a "colored version", we obtain for each $c \in Colors$ that

$$\mathrm{Card}\left(\mathbf{A}_{(c),\lambda}(S_{s,t}(X))\right) \leq \tfrac{3\sqrt{2}}{2} \cdot \tfrac{\ell(S_{s,t}(X))}{\lambda} + 1.$$

Lemma 3 is a direct consequence of the above inequality and the fact that $\ell\,(S_{s,t}(X)) < 1.2k$ with probability $1 - O\left(k^{-\frac{1}{2}}\right)$ according to Corollary 2. The term $\frac{2.55\,k}{\lambda} + 1$ comes from the fact that $1.2 \cdot \frac{3\sqrt{2}}{2} < 2.55$.

We are now prepared to prove Proposition 4. Our proof relies on an adaptation of a result due to Devillers and Hemsley [13, Lemma 7]. The main idea is to discretize the shortest path at different scales and to use standard ideas of (site) percolation theory. Let $\lambda \in 4\mathbb{N}+2$. This restriction on λ ensures the following assertion: for each $v \in \mathbf{G}_{(c)}$, for each $w \in \lambda\mathbf{G}+O_{(c)}$, if the Lebesgue measure of the intersection of the squares $C_2(v)$ and $C_\lambda(w)$ differs from 0, then $C_2(v)$ is included in $C_\lambda(w)$. Taking $\mathcal{E}(c,k,\lambda)$ as in (5), we obtain for each $x > 0$, that

$$\begin{aligned}\mathbb{P}\left[\sharp\mathcal{Y}(\mathbf{A}_{(c)}(S_{s,t}(X))) \geq xk\sqrt{p}\right] = \quad & \mathbb{P}\left[\{\sharp\mathcal{Y}(\mathbf{A}_{(c)}(S_{s,t}(X))) \geq xk\sqrt{p}\} \cap \mathcal{E}(c,k,\lambda)\right] \\ & + \mathbb{P}\left[\{\sharp\mathcal{Y}(\mathbf{A}_{(c)}(S_{s,t}(X))) \geq xk\sqrt{p}\} \cap (\neg\mathcal{E}(c,\lambda,k))\right].\end{aligned}$$

From Lemma 3, the second term equals $O\left(k^{-\frac{1}{2}}\right)$.

The animal $\mathbf{A}_{(c),\lambda}(S_{s,t}(X))$ can be viewed as a sequence of connected squares of size λ, starting at the square containing s. On the event $\mathcal{E}(c,k,\lambda)$, the animal $\mathbf{A}_{(c),\lambda}(S_{s,t}(X))$ belongs to the family $\mathcal{A}_{(c),\lambda}(k)$ of animals $\mathbf{A} \subset \lambda\mathbf{G} + O_{(c)}$ such that $O_{(c)} \in \mathbf{A}$ and $\mathrm{Card}\,(\mathbf{A}) \leq \left\lfloor \frac{2.55k}{\lambda} + 1 \right\rfloor$. Each animal $\mathbf{A} \in \mathcal{A}_{(c),\lambda}(k)$ can be encoded as a word on four letters $\{S, E, N, W\}$ (standing for south, east, north, and west) of length $\mathrm{Card}\,(\mathbf{A})$. Hence, $\mathrm{Card}\left(\mathcal{A}_{(c),\lambda}(k)\right) \leq 4^{\left\lfloor \frac{2.55\,k}{\lambda}+1 \right\rfloor} \leq 4^{\frac{2.55\,k}{\lambda}+1}$. Moreover, since $\lambda \in 4\mathbb{N}+2$, we have

$$\mathbf{A}_{(c)}(S_{s,t}(X)) \subset \mathbf{A}^{(\lambda)}_{(c),\lambda}(S_{s,t}(X)),$$

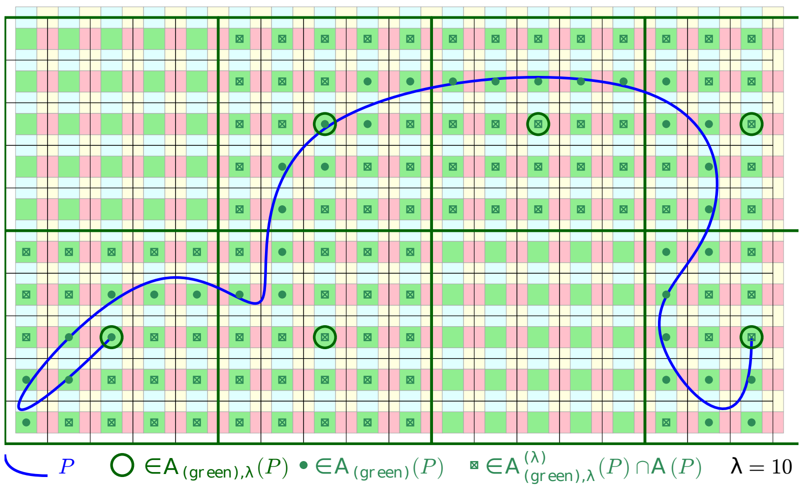

FIGURE 4: The set $\mathbf{A}^{(\lambda)}_{(green),\lambda}(P)$ for a path P .

where, for an animal $\mathbf{A} \subset \lambda\mathbf{G} + O_{(c)}$, we let $\mathbf{A}^{(\lambda)} := \{v \in \mathbf{G}_{(c)} : \exists w \in \mathbf{A}, v \in C_\lambda(w)\}$ (see Figure 4).

This implies that

$$
\begin{aligned}
&\mathbb{P}\left[\{\sharp\mathcal{Y}(\mathbf{A}_{(c)}(S_{s,t}(X))) \geq k\sqrt{p}\} \cap \mathcal{E}(c,k,\lambda)\right] \\
&\quad\leq\quad \mathbb{P}\left[\{\sharp\mathcal{Y}(\mathbf{A}^{(\lambda)}_{(c),\lambda}(S_{s,t}(X))) \geq k\sqrt{p}\} \cap \mathcal{E}(c,k,\lambda)\right] \\
&\quad\leq\quad \mathbb{P}\left[\bigcup_{\mathbf{A}\in\mathcal{A}_{(c),\lambda}(k)} \left\{\sharp\mathcal{Y}(\mathbf{A}^{(\lambda)}) \geq k\sqrt{p}\right\}\right] \\
&\quad\leq\quad \sum_{\mathbf{A}\in\mathcal{A}_{(c),\lambda}(k)} \mathbb{P}\left[\sharp\mathcal{Y}(\mathbf{A}^{(\lambda)}) \geq k\sqrt{p}\right] \\
&\quad\leq\quad \sum_{\mathbf{A}\in\mathcal{A}_{(c),\lambda}(k)} \mathbb{P}\left[\sum_{v\in\mathbf{A}^{(\lambda)}\cap\mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq k\sqrt{p} - 10\right].
\end{aligned}
\tag{6}
$$

The last inequality comes from the fact that $\sum_{v\in\mathbf{A}^{(\lambda)}\cap\mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq \sharp\mathcal{Y}(\mathbf{A}^{(\lambda)}) - 10$ since the number of pixels in $\mathbb{Z}^2$ with color c, which are not in $\mathbb{Z}^{2,(c)}_{s,t}$, is smaller than 10. From the assumption of Proposition 4, we know that for each $\mathbf{A} \in \mathcal{A}_{(c),\lambda}(k)$, the random variable $\sum_{v\in\mathbf{A}^{(\lambda)}\cap\mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]}$ is a binomial distribution with parameters (N,p), where $N := \mathrm{Card}\left(\mathbf{A}^{(\lambda)} \cap \mathbb{Z}^{2,(c)}_{s,t}\right) \leq 0.64\lambda k$ since $\mathrm{Card}\left(\mathbf{A}^{(\lambda)}\right) \leq \frac{2.55k}{\lambda}\frac{\lambda^2}{4} + 1$, $s \in \mathbf{A}^{(\lambda)} \setminus \mathbb{Z}^{2,(c)}_{s,t}$, and $\frac{2.55}{4} \leq 0.64$. Assume that λ is such that

$$k\sqrt{p} - 10 \geq 0.65\lambda kp. \tag{7}$$

Since $k\sqrt{p} - 10 > Np$, it follows from a concentration inequality for a binomial random variable (see e.g. [20, Lemma 1.1]) that for all $\mathbf{A} \in \mathcal{A}_{(c),\lambda}(k)$,

$$\mathbb{P}\left[\sum_{v \in \mathbf{A}^{(\lambda)} \cap \mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq k\sqrt{p} - 10\right] \leq e^{-NpH\left(\frac{k\sqrt{p}-10}{Np}\right)},$$

where $H : (0, \infty) \to [0, \infty)$ is defined by $H(a) = 1 - a + a \log a$ for each $a > 0$. Since, for each $z > 0$, the function $a \mapsto aH\left(\frac{z}{a}\right)$ is decreasing on $[z, \infty)$ and since $Np < 0.64\lambda kp$, we have

$$\mathbb{P}\left[\sum_{v \in \mathbf{A}^{(\lambda)} \cap \mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq k\sqrt{p} - 10\right] \leq e^{-0.64\lambda kpH\left(\frac{k\sqrt{p}-10}{0.64\lambda kp}\right)}.$$

Because H is increasing on $[1, \infty)$, it follows from (7) that

$$\mathbb{P}\left[\sum_{v \in \mathbf{A}^{(\lambda)} \cap \mathbb{Z}^{2,(c)}_{s,t}} \mathbf{1}_{[Y_v]} \geq k\sqrt{p} - 10\right] \leq e^{-0.64\lambda kpH\left(\frac{0.65}{0.64}\right)}.$$

The right-hand side is negligible compared to $k^{-\frac{1}{2}}$ since $\lambda p > 0$ and $H(\frac{0.65}{0.64}) > 0$. It remains to choose λ in such a way that Equation (7) holds. To do it, we take $\lambda \in \left[0, \frac{1.5}{\sqrt{p}}\right] \cap (4\mathbb{N} + 2)$. The fact that such a λ exists is ensured by the condition $p < 0.01$.

4.4.3. *Proof of Proposition 5* To pave the way, we proceed into two steps. First, we choose parameters ε and α in such a way that the strong horizontality property is stronger than the weak horizontality property. Secondly, we provide bounds for the probability that a pixel has a weak horizontality property or an independence property.

Strong vs Weak Horizontality

Lemma 4. *Let $v \in \mathbb{Z}^2$ and $\rho > 0$ and let $\varepsilon_\rho := \sqrt{\rho}\sqrt{2+\rho}$. If the property $\mathcal{H}_\rho(v)$ holds, then $\mathcal{PH}_\rho(v) \subset C^{\varepsilon_\rho}(v)$.*

Proof. Assume that $v = 0$ without loss of generality. Up to a vertical translation, the shortest path between the lines $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ crossing $C(0)$ and intersecting the complement of $C^{\varepsilon_\rho}(0)$ is the segment from $\left(-\frac{1}{2},\frac{1}{2}\right)$ to $\left(\frac{1}{2},\frac{1}{2}+\varepsilon_\rho\right)$. Since the length of this segment is $\sqrt{1+\varepsilon_\rho^2}=1+\rho$, we necessarily have $\mathcal{PH}_\rho(v)\subset C^{\varepsilon_\rho}(v)$.

Lemma 5. *Let $v\in\mathbb{Z}^2$ and $\rho>0$ and $0<\kappa<1$ such that $\frac{\kappa}{\kappa-1}\rho<\frac{\pi^2}{8}$ and let $\alpha_{\rho,\kappa}:=\sqrt{2\frac{\kappa}{\kappa-1}\rho}$. If the property $\mathcal{H}_\rho(v)$ holds then the same is true for $\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)$.*

Proof. We make a proof by contradiction, assuming $\mathcal{H}_\rho(v)$ and $\neg\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)$. The main idea is to split the edges e in $\mathcal{PH}_\rho(v)$ with respect to their angles $|\widehat{e}|$. Indeed,

$$1+\rho\geq\ell\left(\mathcal{PH}_\rho(v)\right)=\sum_{e\in\mathcal{PH}_\rho(v),|\widehat{e}|\leq\alpha_{\rho,\kappa}}\ell\left(e\right)+\sum_{e\in\mathcal{PH}_\rho(v),|\widehat{e}|>\alpha_{\rho,\kappa}}\ell\left(e\right),$$

where the inequality comes from the property $\mathcal{H}_\rho(v)$. For each $e\in\mathcal{PH}_\rho(v)$, we use the trivial inequality $l(e)\geq h(e)$ when $|\widehat{e}|\leq\alpha_{\rho,\kappa}$. If $|\widehat{e}|>\alpha_{\rho,\kappa}$, we notice that, for $\frac{\kappa}{\kappa-1}\rho<\frac{\pi^2}{8}$,

$$l(e)>\frac{h(e)}{\cos\alpha_{\rho,\kappa}}\geq\left(1+\frac{\alpha_{\rho,\kappa}^2}{2}\right)h(e)=\left(1+\frac{\kappa}{\kappa-1}\rho\right)h(e),$$

where the second inequality comes from the fact that $\frac{1}{\cos\alpha}\geq\left(1+\frac{\alpha^2}{2}\right)$ for any $\alpha\in[0,\frac{\pi}{2})$.

It follows that

$$\begin{aligned}1+\rho&>\left(1+\frac{\kappa}{\kappa-1}\rho-\frac{\kappa}{\kappa-1}\rho\right)\sum_{e\in\mathcal{PH}_\rho(v),|\widehat{e}|\leq\alpha_{\rho,\kappa}}h(e)+\left(1+\frac{\kappa}{\kappa-1}\rho\right)\sum_{e\in\mathcal{PH}_\rho(v),|\widehat{e}|>\alpha_{\rho,\kappa}}h(e)\\&=\left(1+\frac{\kappa}{\kappa-1}\rho\right)\sum_{e\in\mathcal{PH}_\rho(v)}h(e)\quad-\frac{\kappa}{\kappa-1}\rho\sum_{e\in\mathcal{PH}_\rho(v),|\widehat{e}|\leq\alpha_{\rho,\kappa}}h(e).\end{aligned}$$

By assumption, the property $\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)$ does not hold. Then we deduce from Lemma 4 that

$$1+\rho>\left(1+\frac{\kappa}{\kappa-1}\rho\right)-\frac{\kappa}{\kappa-1}\rho L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)>1+\frac{\kappa}{\kappa-1}\rho-\frac{\kappa}{\kappa-1}\rho\frac{1}{\kappa}=1+\rho,$$

getting a contradiction.

Pixel Probabilities

Probability for the Independence Property First, we provide an upper bound for the probability that a pixel does not have the independence property.

Lemma 6. *Let $v \in \mathbb{Z}^2_{s,t}$ and let $\varepsilon < \frac{1}{700}$ be fixed. Then*

$$\mathbb{P}\left[\neg \mathcal{I}_\varepsilon(v)\right] \leq 95n^3 e^{-0.194n} + \left(19n^2 + 13n + 4\right) e^{-n\pi}.$$

In the above lemma, we have assumed that $\varepsilon < \frac{1}{700}$ to obtain an upper bound for $\mathbb{P}\left[\neg \mathcal{I}_\varepsilon(v)\right]$ which is independent of ε.

Proof. Let $N_\varepsilon(v)$ be the number of Delaunay triangles in $\mathrm{Del}(X_n)$ such that the associated circumdisk intersects simultaneously $C^\varepsilon(v)$ and the complement of $C_2(v)$. If the event $\mathcal{I}_\varepsilon(v)$ does not hold, then $N_\varepsilon(v) \geq 1$ (here we have used the fact that $\mathcal{I}_\varepsilon(v)$ is $\sigma(X_n \cap C(v))$ measurable). It follows from the Markov's inequality that

$$\mathbb{P}\left[\neg \mathcal{I}_\varepsilon(v)\right] \leq \mathbb{P}\left[N_\varepsilon(v) \geq 1\right] \leq \mathbb{E}\left[N_\varepsilon(v)\right].$$

Besides,

$$\mathbb{E}\left[N_\varepsilon(v)\right] = \frac{1}{3!}\mathbb{E}\left[\sum_{p_{1 \neq .3} \in X_n^3} \mathbf{1}_{[\Delta(p_{1..3}) \in \mathrm{Del}(X_n)]} \mathbf{1}_{[B(p_{1..3}) \not\subset C_2(v);\ B(p_{1..3}) \cap C^\varepsilon(v) \neq \varnothing]}\right].$$

It follows from the Slivnyak-Mecke and the Blaschke-Petkantschin formulas that

$$\begin{aligned}
\mathbb{E}\left[N_\varepsilon(v)\right] &= \frac{n^3}{3!}\int_{(\mathbb{R}^2)^3} \mathbb{P}\left[B(z,r) \cap X_n = \varnothing\right] \mathbf{1}_{[B(z,r) \not\subset C_2(v);\ B(z,r) \cap C^\varepsilon(v) \neq \varnothing]} \mathrm{d}p_{1..3} \\
&= \frac{n^3}{6}\int_{\mathbb{R}_+}\int_{\mathbb{R}^2}\int_{\mathbb{S}^3} e^{-n\,\mathcal{A}(B(z,r))} \mathbf{1}_{[B(z,r) \not\subset C_2(v);\ B(z,r) \cap C^\varepsilon(v) \neq \varnothing]} \\
&\qquad \times r^3 2\,\mathcal{A}\left(\Delta(u_{1..3})\right) \sigma^3(\mathrm{d}u_{1..3}) \mathrm{d}z\,\mathrm{d}r \\
&= \frac{n^3}{6} 24\pi^2 \int_{\mathbb{R}_+}\int_{\mathbb{R}^2} e^{-n\pi r^2} \mathbf{1}_{[B(z,r) \not\subset C_2(v); z \in C^\varepsilon(v) \oplus B(0,r)]} r^3 \mathrm{d}z\,\mathrm{d}r, \qquad (8)
\end{aligned}$$

where the last line comes from Equation (12) and the fact that $B(z,r) \cap C^\varepsilon(v) \neq \varnothing$ if and only if $z \in C^\varepsilon(v) \oplus B(0,r)$. To deal with $B(z,r) \not\subset C_2(v)$ we consider two cases as follows (see Figure 5):

Case 1: if $r \leq 1$, we use the fact that

$$\begin{cases} B(z,r) \not\subset C_2(v) \\ z \in C^\varepsilon(v) \oplus B(0,r) \end{cases} \Rightarrow z \in [-r - \tfrac{1}{2} - \epsilon, r + \tfrac{1}{2} + \epsilon]^2 \setminus [r-1, 1-r]^2.$$

Notice that we have to choose $r > \frac{1-2\epsilon}{4} > 0.249$ to ensure that the set of the right-hand side is not empty. This set has an area smaller than $4(2r + \varepsilon - \frac{1}{2})(\frac{3}{2} + \varepsilon) < 12.016r$.

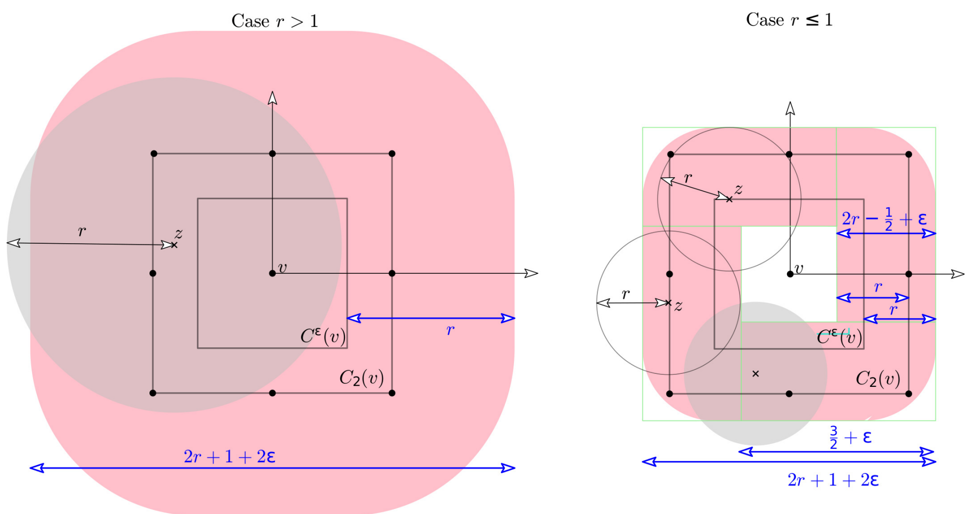

FIGURE 5: The squares $C^{\varepsilon}(v)$, $C_2(v)$ (black), the set $C^{\varepsilon}(v) \oplus B(0,r)$ (pink) and the disk $B(z,r)$ (grey).

Case 2: if $r > 1$, we use the trivial assertion

$$\begin{cases} B(z,r) \not\subset C_2(v) \\ z \in C^{\varepsilon}(v) \oplus B(0,r) \end{cases} \Rightarrow z \in [-r - \tfrac{1}{2} - \epsilon, r + \tfrac{1}{2} + \epsilon]^2.$$

The set of the right-hand side has an area which is lower than $(1+2\varepsilon+2r)^2 < 3.004r^2$. By integrating over z, it follows from (8) that

$$\begin{aligned}
\mathbb{P}[\neg \mathcal{I}_{\varepsilon}(v)] &\leq 4\pi^2 n^3 \left(\int_{0.249}^{1} e^{-n\pi r^2} 12.016\, r^4 \mathrm{d}r + \int_{1}^{\infty} e^{-n\pi r^2} 3.004\, r^5 \mathrm{d}r \right) \\
&\leq 4\pi^2 n^3 \left(e^{-n\pi 0.249^2} \int_{0.249}^{1} 12.016\, r^4 \mathrm{d}r + \int_{1}^{\infty} e^{-n\pi r^2} 3.004\, r^5 dr \right) \\
&\leq 95 n^3 e^{-0.194n} + \left(19n^2 + 13n + 4\right) e^{-n\pi}.
\end{aligned}$$

Probability for the Weak Horizontality Property Secondly, we provide an upper bound for the probability that a pixel has a weak horizontality property conditional on the fact that it has the independence property.

Lemma 7. *Let $v \in \mathbb{Z}^2_{s,t}$ and $\rho < 0.4$. Then there exists $\kappa > 0$ such that $\frac{\kappa}{\kappa-1}\rho < \frac{\pi^2}{8}$ and*

$$\mathbb{P}\left[\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \wedge \mathcal{I}_{\varepsilon_\rho}(v)\right] \leq 31.8 \left(\frac{3}{4} + \frac{\sqrt{\rho}\sqrt{\rho+2}}{2}\right)^2 \sqrt{\rho n},$$

where $\varepsilon_\rho := \sqrt{\rho}\sqrt{2+\rho}$ *and* $\alpha_{\rho,\kappa} := \sqrt{2\frac{\kappa}{\kappa-1}\rho} \in [0, \frac{\pi}{2}]$ *are the same as in Lemma 5.*

Proof. According to the Markov's inequality, we have:

$$\mathbb{P}\left[\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \wedge \mathcal{I}_{\varepsilon_\rho}(v)\right] = \mathbb{P}\left[\{L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \geq \tfrac{1}{\kappa}\} \wedge \mathcal{I}_{\varepsilon_\rho}(v)\right]$$
$$\leq \kappa\, \mathbb{E}\left[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)\mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}\right]. \tag{9}$$

Now, recalling that on the event $\mathcal{I}_{\varepsilon_\rho}(v)$, any triangle, and thus any edge, intersecting $C^{\varepsilon_\rho}(v)$ has its vertices inside $C_2(v)$, we have:

$$\mathbb{E}\left[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)\mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}\right]$$
$$\leq \tfrac{1}{2}\mathbb{E}\left[\sum_{p_{1\neq ..3}\in X_n^3}\mathbf{1}_{[p_{1..3}\in DT(X_n)]}\mathbf{1}_{[|\widehat{p_1p_2}|<\alpha_{\rho,\kappa};\, x_{p_1}\leq x_{p_2}]}\mathbf{1}_{[B(p_{1..3})\subset C_2(v)]}\mathbf{1}_{[B(p_{1..3})\cap C^{\varepsilon_\rho}(v)\neq\varnothing]}h(p_1p_2)\right]$$

where the $\frac{1}{2}$ factor comes from the fact that each edge is counted twice in the sum (once for each incident triangle). We apply the Slivnyak-Mecke and the Blaschke-Petkantschin formulas. This gives:

$$\mathbb{E}\left[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)\mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}\right] \leq \tfrac{1}{2}n^3\int_{C_2(v)}\int_0^1\int_{\mathbb{S}^3} e^{-n\,\mathcal{A}(B(z,r))}\mathbf{1}_{[|\widehat{u_1u_2}|<\alpha_{\rho,\kappa};\, x_{u_1}\leq x_{u_2}]}$$
$$\times\, \mathbf{1}_{[B(z,r)\subset C_2(v)]}\mathbf{1}_{[B(z,r)\cap C^{\varepsilon_\rho}(v)\neq\varnothing]}\cdot r\, h(u_1u_2)\cdot r^3 2\,\mathcal{A}\left(\Delta(u_{1..3})\right)\sigma^3(\mathrm{d}u_{1..3})\mathrm{d}r\mathrm{d}z$$

since $B(z,r)\subset C_2(v)$ implies that $z\in C_2(v)$ and $r\leq 1$.

Hence, $\mathbb{E}\left[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)\mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}\right] \leq a(n,\rho)\times b(\rho,\kappa)$, where

$$a(n,\rho) := \tfrac{1}{2}n^3\int_{C_2(v)}\int_0^1 e^{-n\pi r^2}r^4\mathbf{1}_{[B(z,r)\subset C_2(v)]}\mathbf{1}_{[B(z,r)\cap C^{\varepsilon_\rho}(v)\neq\varnothing]}\mathrm{d}r\mathrm{d}z.$$

$$b(\rho,\kappa) := \int_{\mathbb{S}^3}\mathbf{1}_{[|\widehat{u_1u_2}|<\alpha_{\rho,\kappa};\, x_{u_1}\leq x_{u_2}]}2\,\mathcal{A}\left(\Delta(u_{1..3})\right)h(u_1u_2)\sigma^3(\mathrm{d}u_{1..3}).$$

First, we provide an upper bound for $a(n,\rho)$. Since $B(z,r)\cap C^{\varepsilon_\rho}(v)\neq\varnothing$ we have $z\in v\oplus[-\frac{3}{4}-\frac{\varepsilon_\rho}{2}, \frac{3}{4}+\frac{\varepsilon_\rho}{2}]^2$. Dividing the square $v\oplus[-\frac{3}{4}-\frac{\varepsilon_\rho}{2}, \frac{3}{4}+\frac{\varepsilon_\rho}{2}]^2$ into four quadrants of equal size, it follows that

$$a(n,\rho) \leq \tfrac{1}{2}4n^3\int_0^{\frac{3}{4}+\frac{\varepsilon_\rho}{2}}\int_0^{\frac{3}{4}+\frac{\varepsilon_\rho}{2}}\int_0^\infty e^{-n\pi r^2}r^4\mathrm{d}r\mathrm{d}y_z\mathrm{d}x_z$$
$$= 2n^3\left(\frac{3}{4}+\frac{\varepsilon_\rho}{2}\right)^2\frac{3}{8\pi^2 n^{\frac{5}{2}}}. \tag{10}$$

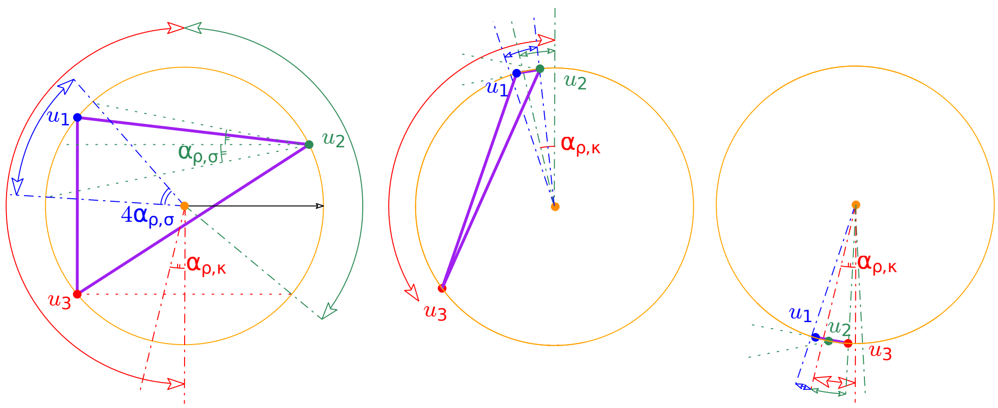

FIGURE 6: Domains of integration for $u(\beta_{1..3})$.

Secondly, to provide an upper bound for $b(\rho,\kappa)$, we write

$$u_{1..3} := u(\beta_{1..3}) := (u(\beta_1), u(\beta_2), u(\beta_3))$$

with $u(\beta_i) = (\cos\beta_i, \sin\beta_i)$. Up to the line symmetry w.r.t. the y-axis, we impose that $\beta_3 \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Up to the line symmetry w.r.t. the x-axis, we also impose that $y_{u(\beta_2)} \geq y_{u_{\beta_3}}$, i.e. $\beta_2 \in [\pi - \beta_3, \beta_3]$. Besides, $u_1 \in C(0,1) \cap \mathcal{C}(\beta_2)$, where $C(0,1)$ is the unit circle and where $\mathcal{C}(\beta_2)$ is the half-cone generated on the left of $u(\beta_2)$ by the x-axis, with vertex $u(\beta_2)$ and with angle $2\alpha_{\rho,\kappa}$, i.e.

$$\mathcal{C}(\beta_2) := u(\beta_2) + \{(r\cos\gamma, r\sin\gamma) : r \geq 0, \gamma \in [\pi - \alpha_{\rho,\kappa}, \pi + \alpha_{\rho,\kappa}]\}.$$

We discuss four cases by splitting the domain of integration $[\pi - \beta_3, \beta_3]$ of β_2 as follows (see Figure 6):

1. If $\beta_2 \in [\pi - \beta_3, \frac{\pi}{2}]$, we have $\beta_1 \in [\pi - \beta_2 - 2\alpha_{\rho,\kappa}, \pi - \beta_2 + 2\alpha_{\rho,\kappa}]$ (Figure 6-left).

2. If $\beta_2 \in [\frac{\pi}{2}, \frac{\pi}{2} + \alpha_{\rho,\kappa}]$, the length of $C(0,1) \cap \mathcal{C}(\beta_2)$ is maximal when $\beta_2 = \frac{\pi}{2}$, so that $\beta_1 \in [\frac{\pi}{2}, \frac{\pi}{2} + 2\alpha_{\rho,\kappa}]$. The area of the triangle $\Delta(u(\beta_{1..3}))$ is less than $2\alpha_{\rho,\kappa}$ and $h(u(\beta_1)u(\beta_2))$ is also less than $2\alpha_{\rho,\kappa}$ (Figure 6-center).

3. If $\beta_2 \in [\frac{\pi}{2} + \alpha_{\rho,\kappa}, \frac{3\pi}{2} - \alpha_{\rho,\kappa}]$, we have $C(0,1) \cap \mathcal{C}(\beta_2) = \varnothing$.

4. If $\beta_2 \in [\frac{3\pi}{2} - \alpha_{\rho,\kappa}, \beta_3]$, with $\beta_3 > \frac{3\pi}{2} - \alpha_{\rho,\kappa}$, the length of $C(0,1) \cap \mathcal{C}(\beta_2)$ is maximal when $\beta_2 = \frac{3\pi}{2}$, so that $\beta_1 \in [\frac{3\pi}{2} - 2\alpha_{\rho,\kappa}, \frac{\pi}{2}]$. The area of the triangle $\Delta(u(\beta_{1..3}))$ is less than $2\alpha_{\rho,\kappa}$ and $h(u(\beta_1)u(\beta_2))$ is also less than $2\alpha_{\rho,\kappa}$ (Figure 6-left).

The integrals associated with the 2^{nd} and 4^{th} cases being bounded by $16\pi\alpha_{\rho,\kappa}^4$, it follows that

$$b(\rho,\kappa) \leq 4\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{\pi-\beta_3}^{\frac{\pi}{2}}\int_{\pi-\beta_2-2\alpha_{\rho,\kappa}}^{\pi-\beta_2+2\alpha_{\rho,\kappa}} 2\,\mathcal{A}\left(\Delta(u(\beta_{1..3}))\right) h(u(\beta_1)u(\beta_2))\mathrm{d}\beta_1\mathrm{d}\beta_2\mathrm{d}\beta_3 + 16\pi\alpha_{\rho,\kappa}^4.$$

Moreover, we know that

$$2\,\mathcal{A}\left(\Delta(u(\beta_{1..3}))\right) = \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix} \quad \text{and} \quad h(u(\beta_1)u(\beta_2)) = \cos\beta_1 - \cos\beta_2.$$

According to Equation (14), this gives:

$$\begin{aligned} b(\rho,\kappa) &\leq \tfrac{256}{9}\cos^3\alpha_{\rho,\kappa}\sin\alpha_{\rho,\kappa} + \tfrac{128}{3}\cos\alpha_{\rho,\kappa}\sin\alpha_{\rho,\kappa} + \tfrac{128}{3}\alpha_{\rho,\kappa} + 16\pi\alpha_{\rho,\kappa}^4 \\ &\leq \tfrac{1024}{9}\alpha_{\rho,\kappa}. \end{aligned}$$

This together with (9), (10) and the fact that $\mathbb{E}\left[L_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v)\mathbf{1}_{[\mathcal{I}_{\varepsilon_\rho}(v)]}\right] \leq a(n,\rho) \times b(\rho,\kappa)$ with $\varepsilon_\rho = \sqrt{\rho}\sqrt{\rho+2}$ and $\alpha_{\rho,\kappa} = \sqrt{2\frac{\kappa}{\kappa-1}\rho}$, implies that

$$\begin{aligned} \mathbb{P}\left[\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \wedge \mathcal{I}_{\varepsilon_\rho}(v)\right] &\leq \kappa\cdot 2n^3\left(\frac{3}{4}+\frac{\sqrt{\rho}\sqrt{\rho+2}}{2}\right)^2 \frac{3}{8\pi^2 n^{\frac{5}{2}}}\cdot\tfrac{1024}{9}\cdot\sqrt{2\frac{\kappa}{\kappa-1}\rho} \\ &= 2\cdot\frac{3}{8\pi^2}\tfrac{1024}{9}\sqrt{2}\left(\frac{3}{4}+\frac{\sqrt{\rho}\sqrt{\rho+2}}{2}\right)^2\sqrt{\tfrac{\kappa^3}{\kappa-1}}\sqrt{\rho n} \\ &\leq 31.76\left(\frac{3}{4}+\frac{\sqrt{\rho}\sqrt{\rho+2}}{2}\right)^2\sqrt{\rho n}. \end{aligned}$$

In the last line, we have taken $\kappa = \frac{3}{2}$ since it is the value of κ which minimizes $\frac{\kappa^3}{\kappa-1}$. The condition $\frac{\kappa}{\kappa-1}\rho < \frac{\pi^2}{8}$ is satisfied since $\rho < \frac{\pi^2}{24} \simeq 0.4$ by assumption.

We can now conclude the proof of Proposition 5. Indeed, such a result is a consequence of Lemmas 5, 6, and 7 and the fact that

$$\mathbb{P}\left[\mathcal{H}_\rho(v) \vee \neg\mathcal{I}_{\varepsilon_\rho}(v)\right] \leq \mathbb{P}\left[\neg\mathcal{I}_{\varepsilon_\rho}(v)\right] + \mathbb{P}\left[\mathcal{H}'_{\varepsilon_\rho,\alpha_{\rho,\kappa},\kappa}(v) \wedge \mathcal{I}_{\varepsilon_\rho}(v)\right].$$

The assumption $\rho < 4\cdot 10^{-6}$ ensures that $\varepsilon_\rho < \frac{1}{700}$ in Lemma 6 and $\rho < 0.4$ in Lemma 7.

5. Concluding Remarks

In this paper, we have provided bounds for the expectation of the length of the shortest path. Experimental values for the expectation $\mathbb{E}\left[\ell\left(P\right)\right]$ and for the standard

path P	$\mathbb{E}[\ell(P)]$	$\sigma(\ell(P))$
$U_{s,t}(X)$	1.1826	0.0053
$S_{s,t}(X)$	1.0401	0.0004

TABLE 1: Experimental values for $U_{s,t}(X)$ and $S_{s,t}(X)$

deviation $\sigma(\ell(P))$ of the lengths of the path $P = U_{s,t}(X), S_{s,t}(X)$ are given in Table 1. These simulations are written with CGAL [8] and are available in [9]. Our estimates are based on 100 trials of Poisson point processes with intensity $n = 10^7$. In particular, Proposition 1 is confirmed by our experiments. However, we notice that the correct value for $\mathbb{E}[\ell(S_{s,t}(X))]$ is about 1.04.

Our lower bound for δ, appearing in Theorem 1, for the expected stretch factor in a Poisson-Delaunay triangulation is far from being tight since the experimental value is much larger. Although several constants in our proof can be improved a little bit, this scheme of proof can only give lower bounds which are far from optimal. Indeed, the first point where our evaluation is quite crude is the approximation of the Strong Horizontality Property by the Weak Horizontality Property. Another point where we widely under-evaluate δ comes from the fact that we use approximation by animals. Actually, a bad situation (when the shortest path is very short) corresponds to an animal with many pixels with a strong horizontality property. However, the converse is clearly not true. Thus we believe that the proof of a tight constant necessitates other techniques.

Acknowledgements

The authors thank David Coeurjolly for pointing out to us ref [17] for Lemma 5, J. Yukich for pointing us ref [25, Theorem 2.2], and Louis Noizet for discussions about the definition of $U_{s,t}(X)$. The authors would like also to thank two referees for their suggestions and comments which improves significantly our paper.

References

[1] FRANZ AURENHAMMER, ROLF KLEIN, DER-TSAI LEE, AND ROLF KLEIN, (2013). Voronoi diagrams and Delaunay triangulations. *vol. 8, World Scientific.*

[2] François Baccelli, Konstantin Tchoumatchenko, and Sergei Zuyev, (2000). Markov paths on the Poisson-Delaunay graph with applications to routing in mobile networks. *Adv. in Appl. Probab.* **32**, no. 1, 1–18.

[3] Prosenjit Bose and Luc Devroye, (2006). On the stabbing number of a random Delaunay triangulation. *Computational Geometry: Theory and Applications* **36**, 89–105.

[4] Prosenjit Bose and Pat Morin, (2004). Online routing in triangulations. *SIAM Journal on Computing* **33**, 937–951.

[5] Pierre Calka, (2002). The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane, *Adv. in Appl. Probab.* **34**, 702–717.

[6] Pedro Machado Manhes de Castro and Olivier Devillers, (2017). Expected length of the Voronoi path in a high dimensional Poisson-Delaunay triangulation, *Discrete and Computational Geometry.*

[7] Frédéric Cazals and Joachim Giesen, (2006). Delaunay triangulation based surface reconstruction. *Effective computational geometry for curves and surfaces, Springer,* 231–276.

[8] Computational Geometry Algorithms Library, urlhttp://www.cgal.org.

[9] Nicolas Chenavier and Olivier Devillers, (2016). Stretch factor of long paths in a planar Poisson-Delaunay triangulation, *Research report 8935,* INRIA.

[10] Siu-Wing Cheng, Tamal K Dey, and Jonathan Shewchuk, (2012). Delaunay mesh generation, *CRC Press.*

[11] J Theodore Cox, Alberto Gandolfi, Philip S Griffin, and Harry Kesten, (1993). Greedy lattice animals I: Upper bounds. *Ann. Appl. Probab.*, 1151–1169.

[12] Olivier Devillers and Louis Noizet, (2016). Walking in a Planar Poisson-Delaunay Triangulation: Shortcuts in the Voronoi Path, *Research Report 8946*, INRIA, August.

[13] Olivier Devillers and Ross Hemsley, (2016). The worst visibility walk in a random Delaunay triangulation is $O(\sqrt{n})$, *Journal of Computational Geometry* **7**, 332-359.

[14] Olivier Devillers, Sylvain Pion, and Monique Teillaud, (2002). Walking in a triangulation, *Internat. J. Found. Comput. Sci.* **13**, 181–199.

[15] Luc Devroye, Christophe Lemaire, and Jean-Michel Moreau, (2004). Expected time analysis for Delaunay point location. *Computational Geometry: Theory and Applications* **29**, 61–89.

[16] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit, (1990). Delaunay graphs are almost as good as complete graphs, *Discrete Comput. Geom.* **5**, 399–407.

[17] Yan Gerard, Antoine Vacavant, and Jean-Marie Favreau, (2015). Tight bounds in the quadtree complexity theorem and the maximal number of pixels crossed by a curve of given length. *Theoretical Computer Science,* 41–55.

[18] Christian Hirsch, David Neuhuser, and Volker Schmidt, (2016). Moderate deviations for shortest-path lengths on random segment process, *ESAIM: Probability and Statistics* **20**, 261-292.

[19] J. Mark Keil and Carl A. Gutwin, (1989). The Delaunay triangulation closely approximates the complete Euclidean graph. *Proc. 1st Workshop Algorithms Data Struct., Lecture Notes Comput. Sci., vol. 382, Springer-Verlag,* 47–56.

[20] M. D. Penrose, (2003). Random Geometric Graphs *Oxford University Press, Oxford.*

[21] Rolf Schneider and Wolfgang Weil, (2008). Stochastic and integral geometry, *Probability and Its Applications, Springer.*

[22] Ge Xia, (2013). The stretch factor of the Delaunay triangulation is less than 1.998, *SIAM J. Comput.* **42**, 1620–1659.

[23] Ge Xia and Liang Zhang, (2011). Toward the tight bound of the stretch factor of Delaunay triangulations. *Proceedings 23th Canadian Conference on Computational Geometry.*

[24] Joseph. E. Yukich, (2015). Surface order scaling in stochastic geometry. *Ann. Appl. Probab.* **25**, 177–210.

[25] Christoph Thäle and Joseph E. Yukich, (2016). Asymptotic theory for statistics of the Poisson-Voronoi approximation. *Bernoulli* **22**, 2372–2400.

Appendix A. Integrals

In this section, we provide values for several integrals which are often used in the paper. These integrals can be computed by using tedious classical computations. These computations are done with Maple (a Maple sheet is available in [9]).

$$\int_0^{\infty} e^{-n\pi r^2} r^k \mathrm{d}r = \frac{\Gamma\left(\frac{k+1}{2}\right)}{2(n\pi)^{\frac{k+1}{2}}}, \quad k > -1; \tag{11}$$

$$\begin{aligned}
&\int_{[0,2\pi)^3} \left| \det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix} \right| \mathrm{d}\beta_{1..3} \\
&\quad = 3! \int_0^{2\pi} \int_0^{\beta_3} \int_0^{\beta_2} \det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix} \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 = 24\pi^2;
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\int_{-1}^{1} \int_{\pi+\arcsin h}^{2\pi-\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} \int_{-\arcsin h}^{\beta_2} \det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix} \\
&\qquad\qquad \times 2\sin\frac{\beta_2-\beta_1}{2} \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \mathrm{d}h \\
&\quad = \frac{35\pi}{3};
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{\pi-\beta_3}^{\frac{\pi}{2}} \int_{\pi-\beta_2-2\alpha_{\rho,\kappa}}^{\pi-\beta_2+2\alpha_{\rho,\kappa}} \det \begin{pmatrix} 1 & 1 & 1 \\ \cos\beta_1 & \cos\beta_2 & \cos\beta_3 \\ \sin\beta_1 & \sin\beta_2 & \sin\beta_3 \end{pmatrix} (\cos\beta_1 - \cos\beta_2) \mathrm{d}\beta_1 \mathrm{d}\beta_2 \mathrm{d}\beta_3 \\
&\quad = \tfrac{64}{9}\cos^3\alpha_{\rho,\kappa}\sin\alpha_{\rho,\kappa} + \tfrac{32}{3}\cos\alpha_{\rho,\kappa}\sin\alpha_{\rho,\kappa} + \tfrac{32}{3}\alpha_{\rho,\kappa}.
\end{aligned} \tag{14}$$